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Z_k-Magic labeling of subdivision graphs

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For any nontrivial abelian group A under addition a graph G is said to be *A-magic* if there exists a labeling $f : E(G) \to A - \{0\}$ such that the vertex labeling f^+ defined as $f^+(v) = \sum f(uv)$ taken over all edges uv incident at v is a constant. An *A-magic* graph G is said to be Z_k-magic graph if the group A is Z_k the group of integers modulo k. These Z_k-magic graphs are referred to as *k-magic* graphs. In this paper, we prove that the graphs such as subdivision of ladder, triangular ladder, shadow, total, flower, generalized prism, $m\Delta_n$-snake, lotus inside a circle, square, gear, closed helm and antiprism are Z_k-magic graphs. Also we prove that if $G_i\,(1 \leq i \leq t)$ be Z_k-magic graphs with magic constant zero then G^t is also Z_k-magic.

Keywords: A-magic labeling; Z_k-magic labeling; Z_k-magic graph; ladder; triangular ladder; shadow; total; flower; generalized prism; $m\Delta_n$-snake; lotus inside a circle; square; gear; closed helm; antiprism.

Mathematics Subject Classification 2010: 05C78

1. Introduction

Graph labeling is currently an emerging area in the research of graph theory. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. A detailed survey was done by Gallian [4]. The concept of an A-magic graph was introduced by Sedlacek [11] as follows: A graph with real-valued edge labeling such that distinct edges have distinct non-negative labels and the sum of the labels of the edges incident to a particular vertex is same for all the vertices. In [13, 14], Stanley noted that Z-magic graphs can be viewed in the more general context of linear homogeneous diophantine equations. Moreover, the generalization of magic graphs and characterization of regular magic graphs are studied by Doop in [1–3]. Lee *et al.* [7–9, 15] studied the construction of magic graphs, V_4-group magic graphs, group magic graphs and group magic Eulerian graphs. For four classical products, Low and Lee [10] examined the A-magic property of the resulting graph obtained from the product of two A-magic graphs. Shiu *et al.* [12] proved that the product and composition of A-magic graphs were A-magic. In [6], Kavitha and Thirusangu obtained the group magic labeling of two cycles with a common vertex. For any nontrivial abelian group A under addition a graph G is said to be A-magic if there exists a labeling $f : E(G) \to A - \{0\}$ such that the vertex labeling f^+ defined as $f^+(v) = \sum f(uv)$ taken over all edges uv incident at v is a constant. An A-magic graph G is said to be Z_k-magic graph if the group A is Z_k the group of integers modulo k. These Z_k-magic graphs are referred to as k-magic graphs. If G is a graph then $S(G)$ is a graph obtained by subdividing each edge of G by a vertex. In [5], Jeyanthi and Jeya Daisy proved that the total graph, square graph, splitting graph, middle graph, $m\Delta_n$-snake graph were Z_k-magic graphs. In this paper, we study the Z_k-magic labeling of some subdivision graphs. Also we prove that if G_i $(1 \leq i \leq t)$ be Z_k-magic graphs with magic constant zero then G^t is also Z_k-magic. We use the following definitions in the subsequent section.

Definition 1.1. The ladder graph L_n is the Cartesian product $P_2 \times P_n$ of a path on two vertices and another path on n vertices.

Definition 1.2. The triangular ladder graph TL_n, $n \geq 2$ is obtained by completing the ladder $P_2 \times P_n$ by adding the edges $v_{1,j}v_{2,j+1}$ for $1 \leq j \leq n$. The vertex set of the ladder is $\{v_{1,j}, v_{2,j} : 1 \leq j \leq n\}$.

Definition 1.3. The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G say G' and G''. Join each vertex u in G' to the neighbors of the corresponding vertex v in G''.

Definition 1.4. The total graph $T(G)$ has the vertex set $V(G) \cup E(G)$ in which two vertices are adjacent whenever they are either adjacent or incident in G.

Definition 1.5. The flower Fl_n is the graph obtained from a helm H_n by joining each pendent vertex to the central vertex of the helm.

Definition 1.6. The graph $C_n \times P_m$ is called the generalized prism.

Definition 1.7. A triangular snake is a connected graph in which all blocks are triangles and the block cut point is a path. Let Δ_n-snake be a triangular snake with n blocks while $m\Delta_n$-snake be a triangular snake with n blocks and every block has m number of triangles with one common edge.

Definition 1.8. Lotus inside a circle LC_n is a graph obtained from the cycle $C_n : u_1, u_2, \ldots, u_n, u_1$ and a star $K_{1,n}$ with the central vertex v_0 and the end vertices $v_1, v_2, \ldots, v_n$ by joining each u_i and $u_{i+1} (\mathrm{mod}\, n)$.

Definition 1.9. For a simple connected graph G, the square of the graph G is denoted by G^2 and is defined as the graph with the same vertex set as of G and two vertices are adjacent in G^2 if they are at a distance 1 or 2 apart in G.

Definition 1.10. The gear graph G_n is obtained from the wheel W_n by adding a vertex between every pair of adjacent vertices in the cycle.

Definition 1.11. The closed helm CH_n is the graph obtained from a helm H_n by joining each pendent vertex to form a cycle.

Definition 1.12. A generalized antiprism A_n^m is obtained by completing the generalized prism $C_m \times P_n$ by adding the edges $v_{i,j}v_{i+1,j}$ for $1 \leq i \leq m$ and $1 \leq j \leq n-1$ where the suffix "i" is taken modulo "m". The vertex set of the generalized prism is $\{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$.

2. Z_k-Magic Labeling of Subdivision Graphs

Theorem 2.1. *The subdivision of ladder graph $S(P_2 \times P_n)$ is Z_k-magic.*

Proof. Let the vertex set and the edge set of $S(P_2 \times P_n)$ be

$$V(S(P_2 \times P_n)) = \{u_i, v_i, w_i : 1 \leq i \leq n\} \cup \{u_i', v_i' : 1 \leq i \leq n-1\}, \quad \text{and}$$

$$E(S(P_2 \times P_n)) = \{u_iu_i', v_iv_i' : 1 \leq i \leq n-1\} \cup \{u_iw_i, w_iv_i : 1 \leq i \leq n-1\} \cup \{u_i'u_{i+1}, v_i'v_{i+1} : 1 \leq i \leq n-1\}.$$

For any integer $a \in \{1, 2, \ldots, \frac{k}{2} - 1\}$ if k is even and $a \in \{1, 2, \ldots, \frac{k-1}{2}\}$ if k is odd. Define the edge labeling $f : E(S(P_2 \times P_n)) \to Z_k - \{0\}$ as follows:

$$f(u_iu_i') = \begin{cases} a & \text{for } i \text{ is odd,} \\ k-a & \text{for } i \text{ is even,} \end{cases}$$

$$f(v_iv_i') = \begin{cases} k-a & \text{for } i \text{ is odd,} \\ a & \text{for } i \text{ is even,} \end{cases}$$

$$f(u_1w_1) = k-a, \quad f(w_1v_1) = a,$$

$$f(u_nw_n) = \begin{cases} k-a & \text{for } i \text{ is odd,} \\ a & \text{for } i \text{ is even,} \end{cases}$$

$$f(w_nv_n) = \begin{cases} a & \text{for } i \text{ is odd,} \\ k-a & \text{for } i \text{ is even.} \end{cases}$$

For $i \neq 1, n$,

$$f(u_iw_i) = \begin{cases} 2a & \text{for } i \text{ is even,} \\ k-2a & \text{for } i \text{ is odd,} \end{cases}$$

For $i \neq 1, n$,

$$f(w_iv_i) = \begin{cases} k-2a & \text{for } i \text{ is even,} \\ 2a & \text{for } i \text{ is odd,} \end{cases}$$

$$f(u_i'u_{i+1}) = \begin{cases} k-a & \text{for } i \text{ is odd,} \\ a & \text{for } i \text{ is even,} \end{cases}$$

$$f(v_i'v_{i+1}) = \begin{cases} a & \text{for } i \text{ is odd,} \\ k-a & \text{for } i \text{ is even.} \end{cases}$$

Then the induced vertex labeling $f^+ : V(S(P_2 \times P_n)) \to Z_k$ is

$$f^+(v) \equiv 0(\bmod k) \quad \text{for all } v \in V(S(P_2 \times P_n)).$$

Hence f^+ is constant and it is equal to $0(\bmod k)$. Since G admits Z_k-magic labeling for any $a \in \{1, 2, \ldots, \frac{k}{2} - 1\}$ if k is even and $a \in \{1, 2, \ldots, \frac{k-1}{2}\}$ if k is odd, the subdivision of ladder graph $S(P_2 \times P_n)$ is a Z_k-magic graph. □

The example for Z_8-magic labeling of $(S(P_2 \times P_5))$ is shown in Fig. 1.

Theorem 2.2. *The subdivision of triangular ladder graph $S(TL_n)$ is Z_k-magic.*

Proof. Let the vertex set and the edge set of $S(TL_n)$ be

$$V(S(TL_n)) = \{u_i, v_i, w_i : 1 \leq i \leq n\} \cup \{u_i', v_i', w_i' : 1 \leq i \leq n-1\}, \quad \text{and}$$
$$E(S(TL_n)) = \{u_iu_i', v_iv_i' : 1 \leq i \leq n-1\} \cup \{u_iw_i, w_iv_i : 1 \leq i \leq n\}$$
$$\cup \{u_i'u_{i+1}, v_i'v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i'w_i', w_i'v_{i+1} : 1 \leq i \leq n-1\}.$$

For any integer $a \in \{1, 2, \ldots, \frac{k}{2} - 1\}$ if k is even and $a \in \{1, 2, \ldots, \frac{k-1}{2}\}$ if k is odd.

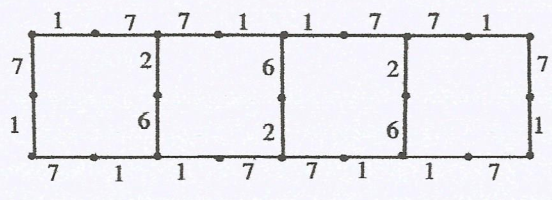

Fig. 1. Z_8-magic labeling of $S(P_2 \times P_5)$.

Define the edge labeling $f: E(S(TL_n)) \to Z_k - \{0\}$ as follows:

$$
\begin{aligned}
&f(u_iu_i') = f(v_iv_i') = a \quad \text{for } 1 \le i \le n-1,\\
&f(u_i'u_{i+1}) = f(v_i'v_{i+1}) = k-a \quad \text{for } 1 \le i \le n-1,\\
&f(u_1w_1) = f(w_nu_n) = a, \quad f(w_1v_1) = f(w_nv_n) = k-a,\\
&f(u_iw_i) = 2a \quad \text{for } 2 \le i \le n-1,\\
&f(w_iv_i) = k-2a \quad \text{for } 2 \le i \le n-1,\\
&f(u_iw_i') = k-2a \quad \text{for } 1 \le i \le n-1,\\
&f(w_i'v_{i+1}) = 2a \quad \text{for } 1 \le i \le n-1.
\end{aligned}
$$

Then the induced vertex labeling $f^+ : V(S(TL_n)) \to Z_k$ is

$$f^+(v) \equiv 0(\text{mod}\, k) \quad \text{for all } v \in V(S(TL_n)).$$

Hence f^+ is constant and it is equal to $0(\text{mod}\, k)$. Since G admits Z_k-magic labeling for any $a \in \{1, 2, \ldots, \frac{k}{2} - 1\}$ if k is even and $a \in \{1, 2, \ldots, \frac{k-1}{2}\}$ if k is odd, the subdivision of triangular ladder graph is a Z_k-magic graph. □

The example for Z_{11}-magic labeling of $S(TL_5)$ is shown in Fig. 2.

Theorem 2.3. *The subdivision of shadow graph $S(D_2(P_n))$ is Z_k-magic.*

Proof. Let the vertex set and the edge set of $S(D_2(P_n))$ be

$$
\begin{aligned}
V(S(D_2(P_n))) &= \{u_i, v_i : 1 \le i \le n\} \cup \{u_i', v_i', w_i, x_i : 1 \le i \le n-1\} \quad \text{and}\\
E(S(D_2(P_n))) &= \{u_iu_i', v_iv_i' : 1 \le i \le n-1\}\\
&\quad \cup \{u_iw_i, w_iv_{i+1}, v_ix_i, x_iu_{i+1} : 1 \le i \le n-1\}\\
&\quad \cup \{u_i'u_{i+1}, v_i'v_{i+1} : 1 \le i \le n-1\}.
\end{aligned}
$$

For any integer $a \in Z_k - \{0\}$.

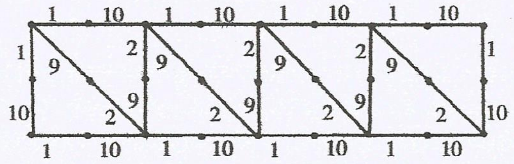

Fig. 2. Z_{11}-magic labeling $S(TL_5)$.

Define the edge labeling $f: E(S(D_2(P_n))) \to Z_k - \{0\}$ as follows:

$$f(u_iu_i') = f(v_iv_i') = a \quad \text{for } 1 \le i \le n,$$
$$f(u_i'u_{i+1}) = f(v_i'v_{i+1}) = k - a \quad \text{for } 1 \le i \le n-1,$$
$$f(u_iw_i) = f(v_ix_i) = k - a \quad \text{for } 2 \le i \le n-1,$$
$$f(w_iv_{i+1}) = f(x_iu_{i+1}) = a \quad \text{for } 1 \le i \le n-1.$$

Then the induced vertex labeling $f^+ : V(S(D_2(P_n))) \to Z_k$ is

$$f^+(v) \equiv 0 (\bmod k) \quad \text{for all } v \in V(S(D_2(P_n))).$$

Hence f^+ is constant and it is equal to $0(\bmod k)$. Since G admits Z_k-magic labeling for any $a \in Z_k - \{0\}$, the subdivision of shadow graph $S(D_2(P_n))$ is a Z_k-magic graph. □

The example for Z_{15}-magic labeling of $S(D_2(P_5))$ is shown in Fig. 3.

Theorem 2.4. *The subdivision of total graph of a path $S(T(P_n))$ is Z_k-magic.*

Proof. Let the vertex set and the edge set of $S(T(P_n))$ be

$$\begin{aligned} V(S(T(P_n))) &= \{u_i : 1 \le i \le n\} \cup \{v_i, w_i, x_i : 1 \le i \le n-1\} \\ &\cup \{u_i' : 1 \le i \le n-1\} \cup \{v_i' : 1 \le i \le n-2\} \quad \text{and} \\ E(S(T(P_n))) &= \{u_iu_i' : 1 \le i \le n-1\} \cup \{v_iv_i' : 1 \le i \le n-2\} \\ &\cup \{u_i'u_{i+1} : 1 \le i \le n-1\} \cup \{v_i'v_{i+1} : 1 \le i \le n-2\} \\ &\cup \{v_iw_i, w_iu_i, v_ix_i, x_iu_{i+1} : 1 \le i \le n-1\}. \end{aligned}$$

For any integer $a \in \{1, 2, \ldots, \frac{k}{2} - 1\}$ if k is even and $a \in \{1, 2, \ldots, \frac{k-1}{2}\}$ if k is odd.

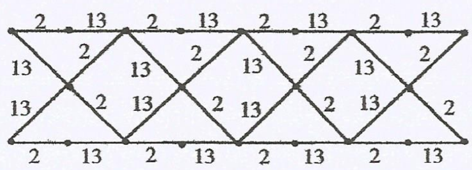

Fig. 3. Z_{15}-magic labeling of $S(D_2(P_5))$.

Define the edge labeling $f : E(S(T(P_n))) \to Z_k - \{0\}$ as follows:

$$f(u_i u_i') = a \quad \text{for } 1 \le i \le n-1,$$

$$f(u_i' u_{i+1}) = k - a \quad \text{for } 1 \le i \le n-1,$$

$$f(v_i v_i') = k - a \quad \text{for } 1 \le i \le n-2,$$

$$f(v_i' v_{i+1}) = 2a \quad \text{for } 1 \le i \le n-2,$$

$$f(v_1 w_1) = 2a, \quad f(w_1 u_1) = k - a,$$

$$f(v_i w_i) = k - a \quad \text{for } 2 \le i \le n-1,$$

$$f(w_i u_i) = a \quad \text{for } 2 \le i \le n-1,$$

$$f(v_i x_i) = a \quad \text{for } 1 \le i \le n-2,$$

$$f(x_i u_{i+1}) = k - a \quad \text{for } 1 \le i \le n-2,$$

$$f(v_{n-1} x_{n-1}) = k - a, \quad f(x_{n-1} u_n) = a.$$

Then the induced vertex labeling $f^+ : V(S(T(P_n))) \to Z_k$ is

$$f^+(v) \equiv 0 (\operatorname{mod} k) \quad \text{for all } v \in V(S(T(P_n))).$$

Hence f^+ is constant and it is equal to $0(\operatorname{mod} k)$. Since G admits Z_k-magic labeling for any $a \in \{1, 2, \ldots, \frac{k}{2} - 1\}$ if k is even and $a \in \{1, 2, \ldots, \frac{k-1}{2}\}$ if k is odd, the subdivision of total graph of a path $S(T(P_n))$ is a Z_k-magic graph. □

The example for Z_5-magic labeling of $S(T(P_6))$ is shown in Fig. 4.

Theorem 2.5. *The subdivision of flower graph $S(Fl_n)$ is Z_k-magic.*

Proof. Let the vertex set and the edge set of $S(Fl_n)$ be

$$V(S(Fl_n)) = \{v, v_i, u_i, w_i, w_i', x_i : 1 \le i \le n\} \quad \text{and}$$

$$E(S(Fl_n)) = \{v w_i, w_i v_i, v_i w_i', w_i' u_i, u_i x_i, x_i v : 1 \le i \le n\}$$
$$\cup \{v_i v_i' : 1 \le i \le n\} \cup \{v_i' v_{i+1} : 1 \le i \le n-1\} \cup \{v_n' v_1\}.$$

For any integer $a \in Z_k - \{0\}$.

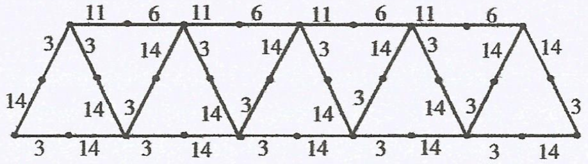

Fig. 4. Z_{17}-magic labeling of $S(T(P_6))$.

Define the edge labeling $f : E(S(Fl_n)) \to Z_k - \{0\}$ as follows:

$$f(vw_i) = f(v_i w_i') = f(u_i x_i) = k - a \quad \text{for } 1 \le i \le n,$$
$$f(w_i v_i) = f(w_i' u_i) = f(x_i v) = a \quad \text{for } 1 \le i \le n,$$
$$f(v_i v_i') = a \quad \text{for } 1 \le i \le n,$$
$$f(v_i' v_{i+1}) = k - a \quad \text{for } 1 \le i \le n-1, \quad f(v_n' v_1) = k - a.$$

Then the induced vertex labeling $f^+ : V(S(Fl_n)) \to Z_k$ is

$$f^+(v) \equiv 0 (\bmod k) \quad \text{for all } v \in V(S(Fl_n)).$$

Hence f^+ is constant and it is equal to $0(\bmod k)$. Since G admits Z_k-magic labeling for any $a \in Z_k - \{0\}$, the subdivision of flower graph $S(Fl_n)$ is a Z_k-magic graph. □

The example of Z_7-magic labeling of $S(Fl_4)$ is shown in Fig. 5.

Theorem 2.6. *The subdivision of generalized prism graph $S(C_m \times P_n)$ is Z_k-magic when m is even.*

Proof. Let the vertex set and the edge set of $S(C_m \times P_n)$ be

$$V(S(C_m \times P_n)) = \{v_i^j, u_i^j : 1 \le i \le m, 1 \le j \le n\} \cup \{w_i^j : 1 \le i \le m, 1 \le j \le n-1\} \quad \text{and}$$

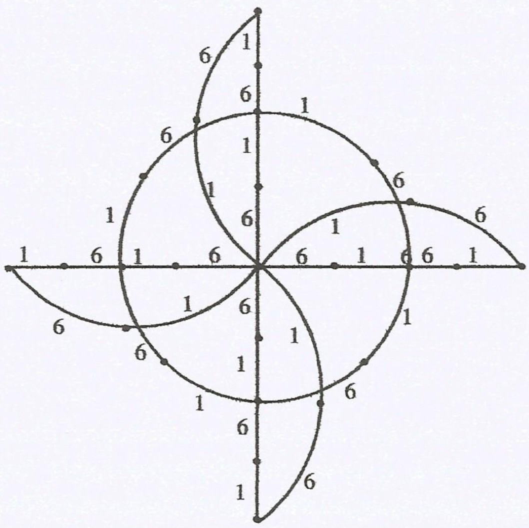

Fig. 5. Z_7-magic labeling of $S(Fl_4)$.

$$\begin{aligned}E(S(C_m \times P_n)) = &\{v_i^j w_i^j, w_i^j v_i^{j+1} : 1 \leq i \leq m, 1 \leq j \leq n-1\}\\ &\cup \{v_i^j u_i^j : 1 \leq i \leq m, 1 \leq j \leq n\}\\ &\cup \{u_i^j v_{i+1}^j : 1 \leq i \leq m-1, 1 \leq j \leq n\}\\ &\cup \{u_m^j v_1^j : 1 \leq j \leq n\}.\end{aligned}$$

For any integer $a \in \{1, 2, \ldots, \frac{k}{2} - 1\}$ if k is even and $a \in \{1, 2, \ldots, \frac{k-1}{2}\}$ if k is odd. Define the edge labeling $f : E(S(C_m \times P_n)) \to Z_k - \{0\}$ as follows:

$$f(v_i^1 u_i^1) = \begin{cases} a & \text{for } i \text{ is odd,} \\ k - a & \text{for } i \text{ is even,} \end{cases}$$

$$f(u_i^1 v_{i+1}^1) = \begin{cases} k - a & \text{for } i \text{ is odd,} \\ a & \text{for } i \text{ is even,} \end{cases}$$

$$f(v_i^j v_i^{j+1}) = k - 2a \quad \text{for } 1 \leq i \leq m, 1 \leq j \leq n-1,$$

$$f(v_i^j w_i^j) = 2a \quad \text{for } 1 \leq i \leq m, 1 \leq j \leq n-1,$$

$$f(v_i^j u_i^j) = a \quad \text{for } 1 \leq i \leq m, 2 \leq j \leq n-1,$$

$$f(u_i^j v_{i+1}^j) = k - a \quad \text{for } 1 \leq i \leq m, 2 \leq j \leq n-1,$$

$$f(v_i^n u_i^n) = \begin{cases} k - a & \text{for } i \text{ is odd,} \\ a & \text{for } i \text{ is even,} \end{cases}$$

$$f(u_i^n v_{i+1}^n) = \begin{cases} a & \text{for } i \text{ is odd,} \\ k - a & \text{for } i \text{ is even.} \end{cases}$$

Then the induced vertex labeling $f^+ : V(S(C_m \times P_n)) \to Z_k$ is

$$f^+(v) \equiv 0 (\bmod k) \quad \text{for all } v \in V(S(C_m \times P_n)).$$

Hence f^+ is constant and it is equal to $0 (\bmod k)$. Since G admits Z_k-magic labeling for any $a \in \{1, 2, \ldots, \frac{k}{2} - 1\}$ if k is even and $a \in \{1, 2, \ldots, \frac{k-1}{2}\}$ if k is odd, the subdivision of generalized prism graph $S(C_m \times P_n)$ is a Z_k-magic graph when m is even. □

The example of Z_{12}-magic labeling of $S(C_4 \times P_5)$ is shown in Fig. 6.

Theorem 2.7. *The subdivision of $m\Delta_n$-snake graph $S(m\Delta_n)$ is Z_k-magic when $k > m$.*

Proof. Let the vertex set and the edge set of $S(m\Delta_n)$-snake be

$$\begin{aligned}V(S(m\Delta_n)) = &\{u_i : 1 \leq i \leq n+1\} \cup \{u_i' : 1 \leq i \leq n\}\\ &\cup \{v_i^j, w_i^j, x_i^j : 1 \leq i \leq n, 1 \leq j \leq m\} \quad \text{and}\end{aligned}$$

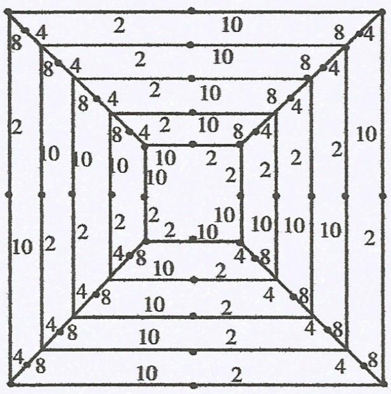

Fig. 6. Z_{12}-magic labeling of $S(C_4 \times P_4)$.

$$E(S(m\Delta_n)) = \{vw_i, w_iv_i, v_iv_i', v_i'u_i, u_iu_i', u_iw_i' : 1 \le i \le n\}$$
$$\cup \{u_i'u_{i+1}, w_i'v_{i+1} : 1 \le i \le n-1\} \cup \{u_n'u_1, w_n'v_1\}.$$

For any integer a such that $ma \in Z_k - \{0\}$.
Define the edge labeling $f : E(S(m\Delta_n)) \to Z_k - \{0\}$ as follows:

$$f(u_iu_i') = ma \quad \text{for } 1 \le i \le n,$$
$$f(u_i'u_{i+1}) = k - ma \quad \text{for } 1 \le i \le n,$$
$$f(u_iw_i^j) = k - a \quad \text{for } 1 \le i \le n, 1 \le j \le m,$$
$$f(w_i^jv_i^j) = a \quad \text{for } 1 \le i \le n, 1 \le j \le m,$$
$$f(v_i^jx_i^j) = k - a \quad \text{for } 1 \le i \le n, 1 \le j \le m,$$
$$f(x_i^ju_{i+1}) = a \quad \text{for } 1 \le i \le n, 1 \le j \le m.$$

Then the induced vertex labeling $f^+ : V(S(m\Delta_n)) \to Z_k$ is

$$f^+(v) \equiv 0 (\operatorname{mod} k) \quad \text{for all } v \in V(S(m\Delta_n)).$$

Hence f^+ is constant and it is equal to $0(\operatorname{mod} k)$. Since G admits Z_k-magic labeling, the subdivision of $m\Delta_n$-snake graph $S(m\Delta_n)$-snake is a Z_k-magic graph when $k > m$. □

The example of Z_5-magic labeling of $S(3\Delta_3)$-snake is shown in Fig. 7.

Theorem 2.8. *The subdivision of lotus inside a circle graph $S(LC_n)$ is Z_k-magic.*

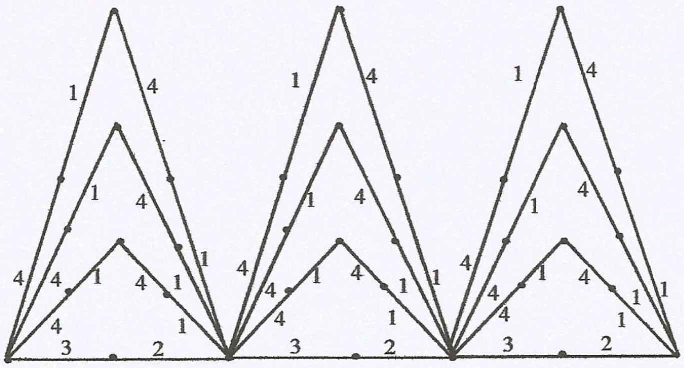

Fig. 7. Z_5-magic labeling of $S(\Delta_3)$-snake.

Proof. Let the vertex set and the edge set of $S(LC_n)$ be

$$V(S(LC_n)) = \{v, v_i, u_i, w_i, v_i', u_i', w_i' : 1 \leq i \leq n\} \quad \text{and}$$

$$E(S(LC_n)) = \{vw_i, w_iv_i, v_iv_i', v_i'u_i, u_iu_i', u_iw_i' : 1 \leq i \leq n\}$$
$$\cup \{u_i'u_{i+1}, w_i'v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_n'u_1, w_n'v_1\}.$$

For any integer $a \in \{1, 2, \ldots, \frac{k}{2} - 1\}$ if k is even and $a \in \{1, 2, \ldots, \frac{k-1}{2}\}$ if k is odd. Define the edge labeling $f : E(S(LC_n)) \to Z_k - \{0\}$ as follows:

$$f(vw_i) = \begin{cases} a & \text{for } i \text{ is odd,} \\ k - a & \text{for } i \text{ is even,} \end{cases}$$

$$f(w_iv_i) = \begin{cases} k - a & \text{for } i \text{ is odd,} \\ a & \text{for } i \text{ is even,} \end{cases}$$

$$f(v_iv_i') = \begin{cases} k - a & \text{for } i \text{ is even,} \\ a & \text{for } i \text{ is odd,} \end{cases}$$

$$f(v_i'u_i) = \begin{cases} a & \text{for } i \text{ is odd,} \\ k - a & \text{for } i \text{ is even,} \end{cases}$$

$$f(u_iu_i') = \begin{cases} k - 2a & \text{for } i \text{ is odd,} \\ a & \text{for } i \text{ is even,} \end{cases}$$

$$f(u_i'u_{i+1}) = \begin{cases} 2a & \text{for } i \text{ is odd,} \\ k - a & \text{for } i \text{ is even,} \end{cases}$$

$$f(u_iw_i') = \begin{cases} 2a & \text{for } i \text{ is odd.} \\ k - 2a & \text{for } i \text{ is even,} \end{cases}$$

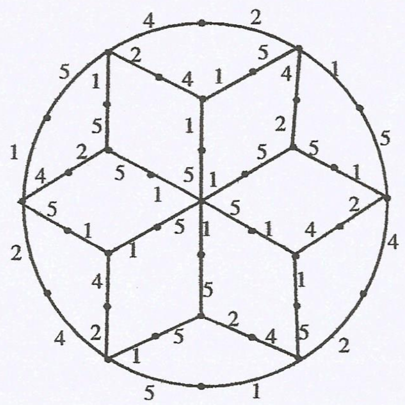

Fig. 8. Z_6-magic labeling of $S(LC_6)$.

$$f(w_i'v_{i+1}) = \begin{cases} k-2a & \text{for } i \text{ is even,} \\ 2a & \text{for } i \text{ is odd,} \end{cases}$$

$$f(u_n'u_1) = k-a, \quad f(w_n'v_1) = 2a.$$

Then the induced vertex labeling $f^+ : V(S(LC_n)) \to Z_k$ is

$$f^+(v) \equiv 0 (\bmod k) \quad \text{for all } v \in V(S(LC_n)).$$

Hence f^+ is constant and it is equal to $0(\bmod k)$. Since G admits Z_k-magic labeling for any integer $a \in \{1, 2, \ldots, \frac{k}{2} - 1\}$ if k is even and $a \in \{1, 2, \ldots, \frac{k-1}{2}\}$ if k is odd, the subdivision of lotus inside a circle graph $S(LC_n)$ is a Z_k-magic graph. □

The example of Z_6-magic labeling of $S(LC_6)$ is shown in Fig. 8.

Theorem 2.9. *The subdivision of square graph of a path $S(T(P_n))$ is Z_k-magic.*

Proof. Let the vertex set and the edge set of $S(P_n^2)$ be

$$V(S(P_n^2)) = \{v_i : 1 \le i \le n\} \cup \{v_i' : 1 \le i \le n-1\} \cup \{w_i : 1 \le i \le n-2\} \quad \text{and}$$

$$E(S(P_n^2)) = \{v_iv_i' : 1 \le i \le n-1\} \cup \{v_i'v_{i+1} : 1 \le i \le n-1\}$$
$$\cup \{v_iw_i : 1 \le i \le n-2\} \cup \{w_iv_{i+2} : 1 \le i \le n-2\}.$$

For any integer $a \in \{1, 2, \ldots, \frac{k}{2} - 1\}$ if k is even and $a \in \{1, 2, \ldots, \frac{k-1}{2}\}$ if k is odd.

Case (i): n is odd.

Define the edge labeling $f : E(S(P_n^2)) \to Z_k - \{0\}$ as follows:

$$f(v_1v_1') = k-a, \quad f(v_1'v_2) = a,$$

$$f(v_iv_i') = a \quad \text{for } 2 \le i \le n-2,$$

$$f(v_i'v_{i+1}) = k - a \quad \text{for } 2 \leq i \leq n-2,$$

$$f(v_{n-1}v_{n-1}') = k - a, \quad f(v_{n-1}'v_n) = a,$$

$$f(v_iw_i) = \begin{cases} a & \text{for } i \text{ is odd,} \\ k-2a & \text{for } i \text{ is even,} \end{cases}$$

$$f(w_iv_{i+2}) = \begin{cases} k-a & \text{for } i \text{ is odd,} \\ 2a & \text{for } i \text{ is even.} \end{cases}$$

Then the induced vertex labeling $f^+ : V(S(P_n^2)) \to Z_k$ is

$$f^+(v) \equiv 0(\text{mod}\, k) \quad \text{for all } v \in V(S(P_n^2)).$$

Case (ii): n is even.

Define the edge labeling $f : E(S(P_n^2)) \to Z_k - \{0\}$ as follows:

$$f(v_1v_1') = k - a, \quad f(v_1'v_2) = a,$$

$$f(v_iv_i') = a \quad \text{for } 2 \leq i \leq n-3,$$

$$f(v_i'v_{i+1}) = k - a \quad \text{for } 2 \leq i \leq n-3,$$

$$f(v_{n-2}v_{n-2}') = k - 2a, \quad f(v_{n-2}'v_{n-1}) = 2a,$$

$$f(v_{n-1}v_{n-1}') = k - a, \quad f(v_{n-1}'v_n) = a,$$

$$f(v_iw_i) = \begin{cases} a & \text{for } i \text{ is odd,} \\ k-2a & \text{for } i \text{ is even,} \quad i \neq n-2, \end{cases}$$

$$f(v_{n-2}w_{n-2}) = a,$$

$$f(w_iv_{i+2}) = \begin{cases} k-a & \text{for } i \text{ is odd,} \\ 2a & \text{for } i \text{ is even,} \quad i \neq n-2, \end{cases}$$

$$f(w_{n-2}v_n) = k - a.$$

Then the induced vertex labeling $f^+ : V(S(P_n^2)) \to Z_k$ is

$$f^+(v) \equiv 0(\text{mod}\, k) \quad \text{for all } v \in V(S(P_n^2).$$

Hence f^+ is constant and it is equal to $0(\text{mod}\, k)$. Since G admits Z_k-magic labeling for any $a \in \{1, 2, \ldots, \frac{k}{2} - 1\}$ if k is even and $a \in \{1, 2, \ldots, \frac{k-1}{2}\}$ if k is odd, the subdivision of square graph of the path $S(P_n^2)$ is a Z_k-magic graph. □

The examples of Z_5-magic labeling of $S(P_7^2)$ and Z_7-magic labeling of $S(P_8^2)$ are shown in Fig. 9.

Theorem 2.10. *The subdivision of gear graph $S(G_n)$ is Z_k-magic when n is even.*

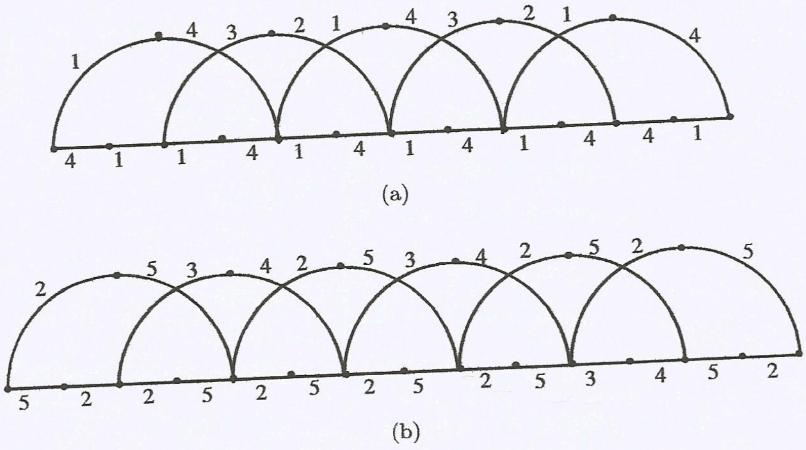

Fig. 9. (a) Z_5-magic labeling of $S(P_7^2)$, (b) Z_7-magic labeling of $S(P_2^8)$.

Proof. Let the vertex set and the edge set of $S(G_n)$ be

$$V(S(G_n)) = \{u, w_i, v_i, v_i', u_i, u_i' : 1 \leq i \leq n\} \quad \text{and}$$

$$E(S(G_n)) = \{uw_i, w_iv_i, v_iv_i', v_i'u_i, u_iu_i' : 1 \leq i \leq n\} \cup \{u_i'v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_n'v_1\}.$$

Case (i): k is even.

For any integer $a \in Z_k - \{0\}$.
Define the edge labeling $f : E(S(G_n)) \to Z_k - \{0\}$ as follows:

$$f(uw_i) = f(w_iv_i) = \frac{k}{2} \quad \text{for } 1 \leq i \leq n,$$

$$f(v_iv_i') = f(u_iu_i') = \begin{cases} a, & \text{for } i \text{ is odd,} \\ \dfrac{k}{2} + a, & \text{for } i \text{ is even,} \end{cases}$$

$$f(v_i'u_i) = f(u_i'v_{i+1}) = \begin{cases} k - a, & \text{for } i \text{ is odd,} \\ \dfrac{k}{2} - a, & \text{for } i \text{ is even.} \end{cases}$$

$$f(u_n'v_1) = \frac{k}{2} - a.$$

Then the induced vertex labeling $f^+ : V(S(G_n)) \to Z_k$ is

$$f^+(v) \equiv 0 (\text{mod}\, k) \quad \text{for all } v \in V(S(G_n)).$$

Case (ii): k is odd.

For any integer $a \in \{1, 3, 5, \ldots, \frac{k+1}{2}\}$.

Define the edge labeling $f : E(S(G_n)) \to Z_k - \{0\}$ as follows:

$$f(uw_i) = \begin{cases} \dfrac{k-a}{2} & \text{for } i \text{ is odd,} \\ \dfrac{k+a}{2} & \text{for } i \text{ is even,} \end{cases}$$

$$f(w_iv_i) = \begin{cases} \dfrac{k+a}{2} & \text{for } i \text{ is odd,} \\ \dfrac{k-a}{2} & \text{for } i \text{ is even,} \end{cases}$$

$$f(v_iv_i') = f(u_iu_i') = \begin{cases} \dfrac{k+a}{2} & \text{for } i \text{ is odd,} \\ a & \text{for } i \text{ is even,} \end{cases}$$

$$f(v_i'u_i) = f(u_i'v_{i+1}) = \begin{cases} \dfrac{k-a}{2} & \text{for } i \text{ is odd,} \\ k-a & \text{for } i \text{ is even.} \end{cases}$$

Then the induced vertex labeling $f^+ : V(S(G_n)) \to Z_k$ is

$$f^+(v) \equiv 0 (\text{mod}\, k) \quad \text{for all } v \in V(S(G_n)).$$

Hence f^+ is constant and it is equal to $0(\text{mod}\, k)$. Since G admits Z_k-magic labeling, the subdivision of gear graph $S(G_n)$ is a Z_k-magic graph when n is even. □

The examples of Z_8- and Z_9-magic labelings of G_4 are shown in Fig. 10.

Theorem 2.11. *The subdivision of closed helm graph $S(CH_n)$ is Z_k-magic when n is even.*

Proof. Let the vertex set and the edge set of $S(CH_n)$ be

$$V(S(CH_n)) = \{v, v_i, u_i, w_i, v_i', u_i', w_i' : 1 \le i \le n\} \quad \text{and}$$

$$E(S(CH_n)) = \{vw_i, w_iv_i, v_iw_i', w_i'u_i, v_iv_i', u_iu_i' : 1 \le i \le n\}$$
$$\cup \{v_i'v_{i+1}, u_i'u_{i+1} : 1 \le i \le n-1\} \cup \{v_n'v_1, u_n'u_1\}.$$

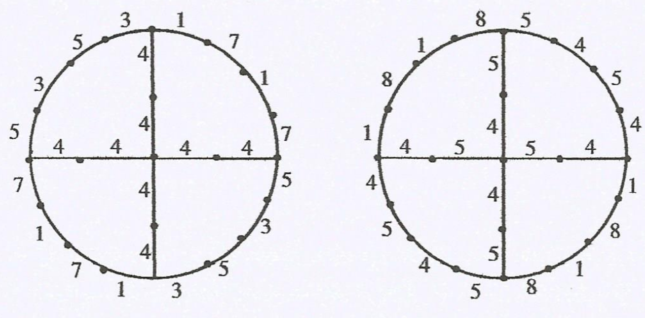

Fig. 10. Z_8- and Z_9-magic labeling of G_4.

For any integer $a \in \{1, 2, \dots, \frac{k}{2} - 1\}$ if k is even and $a \in \{1, 2, \dots, \frac{k-1}{2}\}$ if k is odd. Define the edge labeling $f : E(S(CH_n)) \to Z_k - \{0\}$ as follows:

$$f(vw_i) = \begin{cases} a, & \text{for } i \text{ is odd}, \\ k - a, & \text{for } i \text{ is even}, \end{cases}$$

$$f(w_iv_i) = \begin{cases} k - a, & \text{for } i \text{ is odd}, \\ a, & \text{for } i \text{ is even}, \end{cases}$$

$$f(v_iw_i') = \begin{cases} k - 2a, & \text{for } i \text{ is odd}, \\ 2a, & \text{for } i \text{ is even}, \end{cases}$$

$$f(w_i'u_i) = \begin{cases} 2a, & \text{for } i \text{ is odd}, \\ k - 2a, & \text{for } i \text{ is even}, \end{cases}$$

$$f(v_iv_i') = \begin{cases} a, & \text{for } i \text{ is even}, \\ k - 2a, & \text{for } i \text{ is odd}, \end{cases}$$

$$f(v_i'v_{i+1}) = \begin{cases} k - a, & \text{for } i \text{ is even}, \\ 2a, & \text{for } i \text{ is odd}, \end{cases}$$

$$f(u_iu_i') = \begin{cases} k - a, & \text{for } i \text{ is odd}, \\ 2a, & \text{for } i \text{ is even}, \end{cases}$$

$$f(u_i'u_{i+1}) = \begin{cases} a, & \text{for } i \text{ is odd}, \\ k - a, & \text{for } i \text{ is even}, \end{cases}$$

$$f(v_n'v_1) = 2a, \quad f(u_n'u_1) = k - a.$$

Then the induced vertex labeling $f^+ : V(S(CH_n)) \to Z_k$ is

$$f^+(v) \equiv 0 (\operatorname{mod} k) \quad \text{for all } v \in V(S(CH_n)).$$

Hence f^+ is constant and it is equal to $0(\operatorname{mod} k)$. Since G admits Z_k-magic labeling for any integer $a \in \{1, 2, \dots, \frac{k}{2} - 1\}$ if k is even and $a \in \{1, 2, \dots, \frac{k-1}{2}\}$ if k is odd, the subdivision of closed helm graph $S(CH_n)$ is a Z_k-magic graph when n is even. □

The example of Z_{19}-magic labeling of $S(CH_6)$ is shown in Fig. 11.

Theorem 2.12. *The subdivision of antiprism graph $S(A_m^n)$ is Z_k-magic when m is even.*

Proof. Let the vertex set and the edge set of $S(A_m^n)$ be

$$V(S(A_m^n)) = \{v_{i,j}, w_{i,j}, : 1 \le i \le n, 1 \le j \le m\}$$
$$\cup \{y_{i,j}, x_{i,j} : 1 \le i \le n - 1, 1 \le j \le m\} \quad \text{and}$$

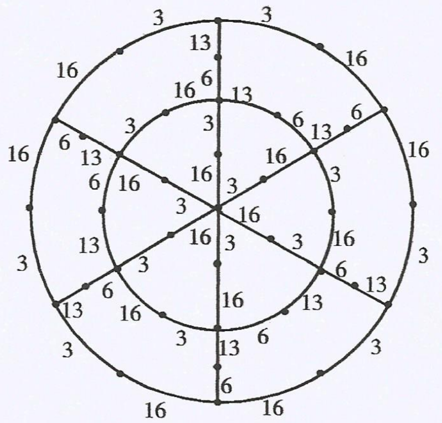

Fig. 11. Z_{10}-magic labeling of $S(CH_6)$.

$$\begin{aligned} E(S(A_m^n)) = {} & \{v_{i,j}w_{i,j} : 1 \le i \le n, 1 \le j \le m\} \\ & \cup \{w_{i,j}v_{i,j+1} : 1 \le i \le n, 1 \le j \le m-1\} \\ & \cup \{w_{i,m}v_{i,1} : 1 \le i \le n\} \cup \{v_{i,j}x_{i,j} : 1 \le i \le n-1, 1 \le j \le m\} \\ & \cup \{x_{i,j}v_{i+1,j} : 1 \le i \le n-1, 1 \le j \le m\} \\ & \cup \{v_{i,j}y_{i,j} : 1 \le i \le n-1, 1 \le j \le m\} \\ & \cup \{y_{i+1,j}v_{i,j+1} : 1 \le i \le n-1, 1 \le j \le m\}. \end{aligned}$$

For any integer $a \in Z_k - \{0\}$.
Define the edge labeling $f : E(S(A_m^n)) \to Z_k - \{0\}$ as follows:

$$\begin{aligned} f(v_{i,j}w_{i,j}) &= a \quad \text{for } 1 \le i \le n, 1 \le j \le m, \\ f(w_{i,j}v_{i,j+1}) &= k-a \quad \text{for } 1 \le i \le n, 1 \le j \le m, \\ f(w_{i,m}v_{i,1}) &= k-a \quad \text{for } 1 \le i \le n, \\ f(v_{i,j}x_{i,j}) &= a \quad \text{for } 1 \le i \le n-1, 1 \le j \le m, \\ f(x_{i,j}v_{i+1,j}) &= k-a \quad \text{for } 1 \le i \le m, 1 \le j \le n-1, \\ f(v_{i,j}y_{i,j}) &= k-a \quad \text{for } 1 \le i \le n-1, 2 \le j \le m, \\ (y_{i+1,j}v_{i,j+1}) &= a \quad \text{for } 1 \le i \le n-1, 2 \le j \le m. \end{aligned}$$

Hence f^+ is constant and it is equal to $0(\text{mod}\, k)$. Since G admits Z_k-magic labeling for any $a \in Z_k - \{0\}$, the subdivision of antiprism $S(A_m^n)$ is a Z_k-magic graph when m is even. □

The example of Z_{10}-magic labeling of $S(A_4^3)$ is shown in Fig. 12.

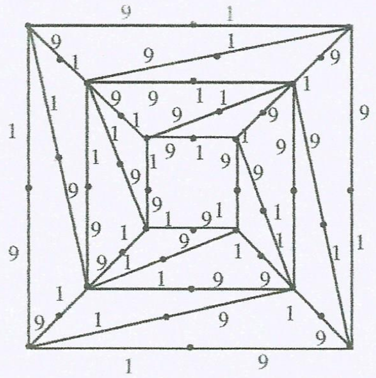

Fig. 12. Z_{10}-magic labeling of A_4^3.

Theorem 2.13. *Let G_i $(1 \le i \le t)$ be Z_k-magic graphs with magic constant zero. Let $u_i \in V(G_i)$ for $1 \le i \le t$. If G^t is a graph obtained by identifying the vertices $u_2, u_3, \ldots, u_t$ with u_1 is also Z_k-magic.*

Proof. Let f_i $(1 \le i \le t)$ be the Z_k-magic labeling of G_i,

$$f_i^+(v) \equiv 0(\bmod k) \quad \text{for all } 1 \le i \le t.$$

Define the edge labeling $g : E(G^t) \to Z_k - \{0\}$ by

$$g(e) = f_i(e) \quad \text{if } e \in E(G_i).$$

Then the induced vertex labeling $g^+ : V(G^t) \to Z_k$ is

$$\begin{aligned} g^+(u) &= f_i^+(u) \quad \text{if } u \in V(G_i) \\ &\equiv 0(\bmod k), \\ g^+(u_1) &= \sum f_i^+(u_i) \quad \text{for } 1 \le i \le t \\ &\equiv 0(\bmod k). \end{aligned}$$

Therefore G^t is a Z_k-magic graph. □

References

[1] M. Doob, On the construction of the magic graphs, in *Proc. Fifth Southeastern Conf. on Combinatorics, Graph Theory and Computing* (1974), pp. 361–374.

[2] M. Doob, Generalizations of magic graphs, *J. Combin. Theory Ser. B* **17** (1974) 205–217.

[3] M. Doob, Characterizations of regular magic graphs, *J. Combin. Theory Ser. B* **17** (1978) 94–104.

[4] J. A. Gallian, A dynamic survey of graph labeling, Electron. *J. Combin.* (2015) #DS6.

[5] P. Jeyanthi and K. Jeya Daisy, Certain classes of Z_k-magic graphs, to appear in *J. Graph Labeling*.

[6] K. Kavitha and K. Thirusangu, Group magic labeling of cycles with a common vertex, *Int. J. Comput. Algorithm* **2** (2013) 239–242.

[7] S. M. Lee, F. Saba, E. Salehi and H. Sun, On the $V4$-group magic graphs, *Congr. Numer.* **156** (2002) 59–67.

[8] S. M. Lee, H. Sun and L. Wen, On group magic graphs, *J. Combin. Comput.* **38** (2001) 197–207.

[9] R. M. Low and S. M. Lee, On group magic Eulerian graphs, *J. Combin. Math. Combin. Comput.* **50** (2004) 141–148.

[10] R. M. Low and S. M. Lee, On the products of group-magic graphs, *Australas. J. Combin.* **34** (2006) 41–48.

[11] J. Sedlacek, On magic graphs, *Math. Slov.* **26** (1976) 329–335.

[12] W. C. Shiu, P. C. B. Lam and P. K. Sun, Construction of magic graphs and some A-magic graphs with A of even order, *Cong. Numer.* **167** (2004) 97–107.

[13] R. P. Stanley, Linear homogeneous Diophantine equations and magic labelings of graphs, *Duke Math. J.* **40** (1973) 607–632.

[14] R. P. Stanley, Magic labelings of graphs, symmetric magic squares, systems of parameters and Cohen–Macaulay rings, *Duke Math. J.* **43** (1976) 511–531.

[15] G. W. Sun and S. M. Lee, Construction of magic graphs, *Congr. Numer.* **103** (1994) 243–251.